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Shear flow instability in a conducting viscous fluid

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The effect of a parallel magnetic field upon the stability of the plane interface between two conducting viscous fluids in uniform relative motion is considered. A parameter reduction, which has not previously been noted, is employed to facilitate the solution of the problem. Neutral stability curves for unrestricted ranges of the governing parameters are found, and the approximate solutions of other authors are examined in this light.

1. Introduction

The problem of the stability of the plane interface between two fluids in relative motion has received considerable attention. Of recent interest is the question of the effect of a magnetic field upon the stability of such flows. In an attempt to furnish an answer to this question, Gotoh (1961, 1971) and Abas (1969) considered the effect of a parallel magnetic field on the stability of flows of the free boundary-layer type. However, both these authors made certain simplifying assumptions in their treatments, and therefore provide only incomplete solutions.

In this paper we shall consider the stability of a shear layer in the presence of a magnetic field (parallel to the basic flow), but without recourse to the approximations employed by other authors. Essentially the problem is a three-parameter one involving the Reynolds number R, the magnetic Reynolds R_m and the Alfvén number A. Both Abas and Gotoh used the 'small R_m' approximation considered by Stuart (1954), which leads to a reduction in the number of parameters: R_m and A^2 combine to give the single magnetic interaction number A^2R_m. However, we shall show that a simple transformation, valid for the Helmholtz profile, leads to a reduction in the number of parameters, without any approximation being necessary.

2. The fluid equations

We consider an incompressible fluid of uniform conductivity σ, magnetic permeability μ, density ρ and kinematic viscosity ν. In a Cartesian co-ordinate system (x_1, x_2, x_3) the steady state we wish to consider is

$$\left.\begin{aligned} \mathbf{v} = \mathbf{v}_0 = (v_0, 0, 0), &\quad \text{where} \quad v_0 = v_0(x_2),\\ \mathbf{B} = \mathbf{B}_0 = (B_0, 0, 0), &\quad B_0 = \text{constant}, \end{aligned}\right\} \tag{2.1}$$

where $\mathbf{v}$ denotes the fluid velocity and $\mathbf{B}$ the magnetic induction field. For the basic state to satisfy the equations of motion it is necessary that the velocity distribution be parabolic. However, it will be assumed that a wider class of flows (e.g. the free boundary layer between parallel streams) may be included in our discussion. This matter is discussed in further detail in Lin (1955 pp. 52,115).

To describe perturbations of this basic state we put

$$\mathbf{v}_1 = (v_1(x_2), v_2(x_2), v_3(x_2)) \exp(-i\alpha_1 \omega t + i\alpha_1 x_1 + i\alpha_3 x_3),$$

$$\mathbf{b}_1 = (b_1(x_2), b_2(x_2), b_3(x_2)) \exp(-i\alpha_1 \omega t + i\alpha_1 x_1 + i\alpha_3 x_3),$$

where $\mathbf{v}_1$ denotes the perturbation velocity and $\mathbf{b}_1$ the perturbation magnetic induction field. Here α_1 and α_3 are positive constants, and ω is complex. Then it can be shown (Stuart 1954) that v and ψ satisfy the (dimensionless) equations

$$(U-c)\psi - v = (i\alpha R_m)^{-1}(\psi'' - \lambda^2\psi), \tag{2.2}$$

$$(U-c)(v''-\lambda^2 v) - U''v - A^2(\psi''-\lambda^2\psi) = (i\alpha R)^{-1}(v^{\text{iv}} - 2\lambda^2 v'' + \lambda^4 v), \tag{2.3}$$

where $\quad v_2 = Vv, \quad v_0 = VU(y), \quad b_2 = B_0\psi, \quad x_2 = Ly, \quad \alpha = L\alpha_1,$

$$\lambda = L(\alpha_1^2 + \alpha_3^2)^{\frac{1}{2}}, \quad R = LV/\nu, \quad R_m = 4\pi\mu\sigma LV, \quad A^2 = B_0^2/4\pi\mu\rho V^2.$$

Here L and V are characteristic length and velocity scales (respectively), and a prime denotes differentiation with respect to y.

We consider in detail the case of a steady-state velocity profile

$$U(y) = \begin{Bmatrix} 1, & y > 0, \\ -1, & y < 0. \end{Bmatrix} \tag{2.4}$$

The use of such a discontinuous profile may be justified in the case of a long wave (i.e. small wavenumber); our results are only physically realistic for long waves. A discussion of the use of discontinuous profiles in relation to the Orr–Sommerfeld equation has been given by Drazin (1961).

Following Drazin we derive the boundary conditions that pertain at the plane $y = 0$ by examining certain integrals of equations (2.2) and (2.3). Such a process leads to the conditions (for the profile (2.4))

$$[\psi] = [\psi'] = 0, \quad [v] = [v'] = [v'' + i\alpha R(U-c)v] = [v''' - i\alpha R(U-c)v'] = 0, \tag{2.5}$$

where the notation $[\psi] \equiv \psi(0_+) - \psi(0_-)$ has been used. Finally, conditions (2.5) must be supplemented by the requirement that the disturbance vanishes at infinity:

$$v, \psi \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.6}$$

3. The transformation

To determine the stability nature of the state (2.1) we solve (2.2) and (2.3) subject to suitable boundary conditions. This procedure leads to a dispersion relationship of the form

$$F(c, \lambda, \alpha R, \alpha R_m, A^2) = 0, \tag{3.1}$$

which determines c in terms of the four parameters λ, αR, αR_m and A^2. The large number of free parameters in this relationship has led a number of authors to make certain simplifying assumptions. For example, Gotoh (1961) restricted his analysis to the case where R and R_m are either both large or both small, and both Abas (1969) and Gotoh (1971) treated the case for which $R_m \ll 1$ (see also Stuart 1954). When $R_m \ll 1$, c is determined by the parameters λ, αR and $\alpha A^2 R_m$.

However, for the profile (2.4) a parameter reduction in (3.1) is possible without such assumptions being made. We consider equations (2.2) and (2.3) and put

$$\beta = \lambda/\bar{R}, \quad \bar{y} = \bar{R}y \quad (0 < \bar{R} < \infty), \tag{3.2}$$

where $\bar{R}$ is defined by the Squire transformation (Squire 1933)

$$\bar{R} = \alpha R/\lambda.$$

Then, under the transformation (3.2), equations (2.2) and (2.3) become

$$(U-c)\,\psi - v = (i\beta P_m)^{-1}(D^2-\beta^2)\,\psi, \tag{3.3}$$

$$(U-c)(D^2-\beta^2)\,v-(D^2U)\,v-A^2(D^2-\beta^2)\,\psi = (i\beta)^{-1}(D^4-2\beta^2D^2+\beta^4)\,v, \tag{3.4}$$

where $D \equiv d/d\bar{y}$, and $P_m = R_m/R$ is the magnetic Prandtl number. Also, under the transformation, (3.1) becomes

$$G(c, \beta, P_m, A^2) = 0.$$

Clearly, c is now determined by only three parameters, namely β, P_m and A^2.

Finally, we note that the transformation (3.2) is not useful in bounded flows because then the boundary conditions are functions of R.

4. The shear flow

Our problem is to solve (3.3) and (3.4) subject to conditions (2.5) and (2.6). For the profile (2.4), equations (3.3) and (3.4) are simply ordinary differential equations with constant coefficients, and as such possess exponential-type solutions. If we eliminate v between (3.3) and (3.4) then the resulting equation demands that ψ be of the form

$$\psi = \begin{cases} A_1 e^{-\beta\bar{y}} + A_2 e^{-m_2\bar{y}} + A_3 e^{-m_3\bar{y}} & (\bar{y} > 0), \\ B_1 e^{\beta\bar{y}} + B_2 e^{n_2\bar{y}} + B_3 e^{n_3\bar{y}} & (\bar{y} < 0), \end{cases} \tag{4.1}$$

where

$$m_2 = [\beta^2 + \tfrac{1}{2}i\beta\{(1+P_m)(1-c)+q_1\}]^{\frac{1}{2}}, \quad n_2 = [\beta^2 + \tfrac{1}{2}i\beta\{(1+P_m)(-1-c)-q_2\}]^{\frac{1}{2}},$$

$$m_3 = [\beta^2 + \tfrac{1}{2}i\beta\{(1+P_m)(1-c)-q_1\}]^{\frac{1}{2}}, \quad n_3 = [\beta^2 + \tfrac{1}{2}i\beta\{(1+P_m)(-1-c)+q_2\}]^{\frac{1}{2}},$$

$$q_1 = [4A^2P_m + (1-P_m)^2(1-c)^2]^{\frac{1}{2}}, \qquad q_2 = [4A^2P_m + (1-P_m)^2(1+c)^2]^{\frac{1}{2}}.$$

Here condition (2.6) has been applied, so in taking square roots the convention of selecting the root with positive real part is used. The constants $A_1, \ldots, B_3$ are determined by the remaining boundary conditions.

Application of the conditions (2.5), rewritten purely in terms of ψ, leads to a system of six algebraic equations which may be conveniently written in the form

$$\mathbf{AX} = 0, \tag{4.2}$$

where $\mathbf{X}$ is a 6×1 column vector consisting of the elements $A_1, \ldots, B_3$. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ \beta & m_2 & m_3 & \beta & n_2 & n_3 \\ \beta^2-2i\beta P_m & m_2^2-2i\beta P_m & m_3^2-2i\beta P_m & -\beta^2 & -n_2^2 & -n_3^2 \\ \beta^3-2i\beta^2 P_m & m_2^3-2im_2\beta P_m & m_3^3-2im_3\beta P_m & \beta^3 & n_2^3 & n_3^3 \\ \beta^4-r_1-\beta^2 s_1 & m_2^4-r_1-m_2^2 s_1 & m_3^4-r_1-m_3^2 s_1 & -\beta^4 & -n_2^4 & -n_3^4 \\ \beta(\beta^4-r_2-\beta^2 s_2) & m_2(m_2^4-r_2-m_2^2 s_2) & m_3(m_3^4-r_2-m_3^2 s_2) & \beta^5 & n_2^5 & n_3^5 \end{pmatrix} \quad (4.3)$$

Here

$$r_1 = 2i\beta^2(P_m+1)(\beta-iP_m c)+2\beta^2 P_m(P_m-1), \quad s_1 = 2i\beta(P_m-1),$$

$$r_2 = 2i\beta^2(P_m-1)(\beta-iP_m c)+2\beta^2 P_m(P_m+1), \quad s_2 = 2i\beta(P_m+1).$$

The condition for a non-trivial solution of (4.3) yields the dispersion equation

$$\det(\mathbf{A}) = 0, \quad (4.4)$$

which determines the complex wave speed c in terms of β, P_m and A^2. Equation (4.4) may be written in the form

$$\beta(m_2-\beta)(m_3-\beta)(n_2-\beta)(n_3-\beta)(m_3-m_2)(n_3-n_2)\det(\mathbf{C}) = 0,$$

where $\mathbf{C}$ is a 4×4 matrix, the elements of which will not be given. Discarding the solutions arising from the (above) product of factors,† we consider the roots arising from the equation

$$\det(\mathbf{C}) = 0. \quad (4.5)$$

5. Neutral stability: results and discussion

Neutral disturbances are characterized by the condition $c_i = 0$ ($c = c_r + ic_i$). However, Tatsumi & Gotoh (1960) have shown that for profiles of the type (2.4) $c_r \equiv 0$ (provided that (3.3) and (3.4) possess a unique solution). In the absence of a magnetic field, this has been demonstrated explicitly (Drazin 1961). Therefore, to investigate the neutral solutions of (4.5) we shall take c to be zero.

An iterative method based upon the Newton–Raphson procedure was used to find the roots (i.e. the values of β for fixed P_m and A^2) of (4.5). The results of this computation are shown in figures 1–5.

For convenience we have presented our results for the case of a two-dimensional (i.e. $\lambda = \alpha$) disturbance. In the absence of a magnetic field the growth rate of a three-dimensional wave is the same as that of a two-dimensional wave at a lower Reynolds number (see Squire 1933). However, when a field is introduced this is not necessarily the case (Hunt 1966), and so some comment on three-dimensional disturbances is necessary. In fact, it is clear from the analysis of the

† These solutions correspond to the occurrence of terms like $\bar{y}\exp(\beta\bar{y})$ in (4.4) and therefore warrant a separate treatment. In the absence of a magnetic field such roots lead to null eigenvectors.

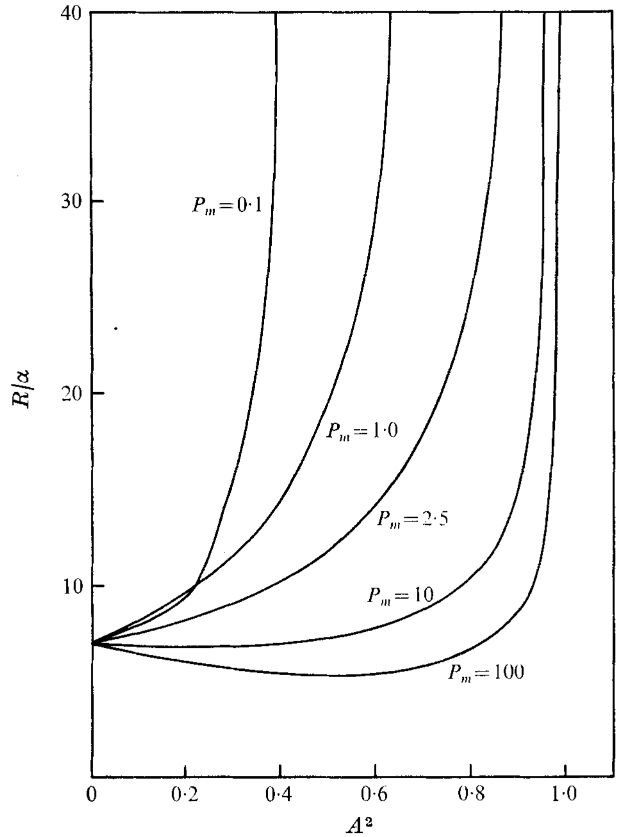

FIGURE 1. Neutral stability curves for various P_m. Instability is the region between the curve and the ordinate R/α.

previous sections that figures 1–5 apply to three-dimensional disturbances provided that R/α is replaced by $\alpha R/\lambda^2 = (R/\alpha)\cos^2\theta$, $0 \leqslant \theta < \frac{1}{2}\pi$, where θ is the angle the direction of propagation of the disturbance makes with the basic flow.

Figures 1–3 show neutral stability curves for various values of the magnetic Prandtl number P_m. The cases $P_m \gtrless 1$ exhibit quite different features and have therefore been presented separately (figures 1 and 2); for comparison the case $P_m = 0{\cdot}1$ is given in all three figures.

It is clear from figure 1 that for large P_m, say $P_m \geqslant 10$, the neutral curves appear to tend asymptotically to the line $A = 1$. Thus, for a fixed $P_m \geqslant 1$ the instability is completely suppressed for $A > 1$. For $A < 1$ the instability is always present for some value of R/α. Some caution must be used in interpreting figure 1. It does not imply that for a given $A > 1$ and given R_m the disturbance is stable, but rather than for a given $A > 1$ and a given R_m the disturbance is stable provided that R is sufficiently small so that $P_m > P_{m_0}$, for some P_{m_0}. The critical value P_{m_0} has not been determined precisely, though from figures 1 and 2 it is clear that $0{\cdot}02 < P_{m_0} < 0{\cdot}1$.

For an ideally conducting viscous fluid ($P_m \to \infty$) it follows from figure 1 that the critical value of the Alfvén number is unity. This is in agreement with the

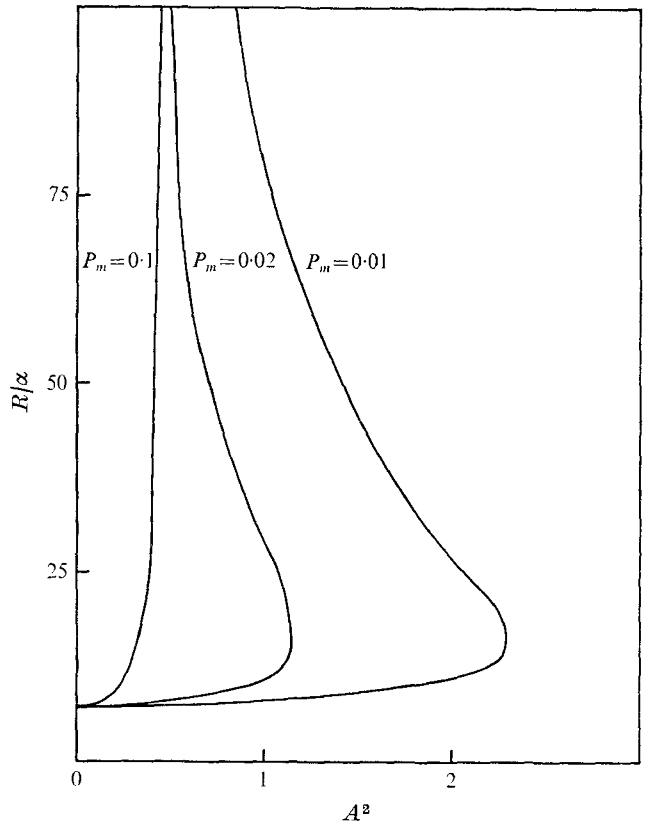

FIGURE 2. Neutral stability curves for various P_m ($P_m < 1$). Instability is the region between the curve and the ordinate R/α.

sufficiency condition for stability (of the flow of an ideally conducting fluid) given by Velikhov (1959).

Figure 2 shows how events develop for values of P_m less than 1. For $P_m < P_{m_0}$, decreasing P_m increases the region of instability: the smaller the value of P_m, the larger A must be to suppress the instability.

In contrast to figure 1 the line $A = 1$ plays no significant role in figure 2. Also in contrast to figure 1 is the 'two-root feature' that was commented upon by Abas (1969) and overlooked by Gotoh (1961). This feature is evident in figure 2 for the cases $P_m = 0{\cdot}01$ and $P_m = 0{\cdot}02$, though it is absent at $P_m = 0{\cdot}1$. Abas's results are for the case when $P_m \ll 1$, whereas our results hold more generally, having been derived without the use of this approximation.

For small P_m (say $P_m \leqslant 0{\cdot}01$) good agreement exists between Abas's approximate analysis and our treatment. On this basis, therefore, we may conclude that for $P_m \leqslant 0{\cdot}01$ the magnetic field suppresses the instability if $A > A_{\text{crit}}$, where $A_{\text{crit}} \doteqdot (0{\cdot}0233)/P_m$. The approximate value 0·0233 is the critical value† of A^2R_m/R found by Abas.

† Gotoh (1971) suggests that this critical value is in fact higher, and gives the value 0·0295.

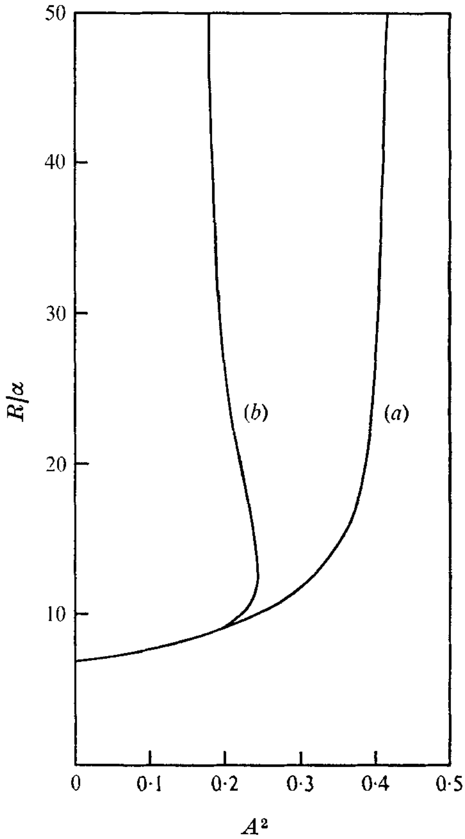

FIGURE 3. Neutral stability curves for $P_m = 0{\cdot}1$. (*a*) General P_m results. (*b*) Calculations by Abas.

Though for $P_m \leqslant 0{\cdot}01$ good agreement exists between Abas's results and ours, this is not, of course, the case for larger values of P_m. To trace the growth of this disparity we have plotted in figure 3 our neutral stability curve, marked (*a*), for the case $P_m = 0{\cdot}1$. For comparison with curve (*a*) we have replotted the results of Abas (marked as curve (*b*)).

For values of A^2 less than about 0·2 and values of R/α about 10 good agreement exists between curves (*a*) and (*b*), and also with the approximate formula (Gotoh 1961)

$$R/\alpha = 4 \times 3^{\frac{1}{2}}(1 + 13A^2R_m/R),$$

valid for $R_m \ll R$, $A^2R_m \ll R$. At about $A^2 = 0{\cdot}22$ the curves (*a*) and (*b*) diverge. We may conclude from this (and figures 1 and 2) that the 'two-root feature' is characteristic of our results for small P_m, and *not* a product of the 'small R_m' approximation.

The intermediate range of magnetic Prandtl number $0{\cdot}1 < P_m < 1$ is not directly covered by figures 1 and 2. In this region the change-over from the behaviour of the curves for $P_m \geqslant 1$ to the quite different behaviour illustrated in figure 2 occurs. There is no sharp division: the 'two-root feature' develops gradually and without discontinuity.

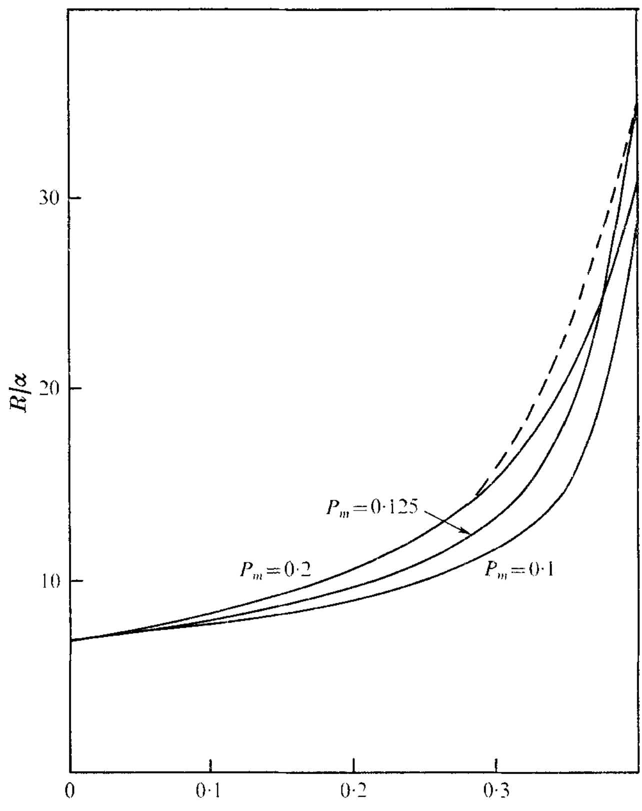

FIGURE 4. Neutral stability curves in the range $0{\cdot}1 \leqslant P_m < 1$. ——, neutral curves; – – – –, conjectured bound on the neutral curves.

The shift of the neutral curves towards the line $A = 0$ (as indicated in figure 1 for P_m decreasing and in figure 2 for P_m increasing) appears to stop at $P_m \doteqdot 0{\cdot}125$, this value being computed from the behaviour of the neutral curves for large R/α. Thus the curve $P_m = 0{\cdot}125$ provides an approximate lower bound on the family of neutral curves as $R/\alpha \to \infty$, the bound being based on the A^2 axis. This behaviour is illustrated in figure 4 for two values of P_m, one larger and one smaller than the 'critical' value $P_m = 0{\cdot}125$. For values of P_m closer to $1{\cdot}0$, such as $P_m = 0{\cdot}75$ and $P_m = 0{\cdot}5$, the trend indicated in figure 1 by the curves for $P_m \geqslant 1{\cdot}0$ is maintained. Note from figure 4 that the 'critical curve' does not provide a complete bound upon the neutral curves, and in fact no neutral curve is such a bound. Of course, given a fixed value of A the bounding curve (at that value) may be readily found. For example, with the values $A^2 = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ the bounding neutral curves are $P_m \doteqdot 0{\cdot}34$, $0{\cdot}29$ and $0{\cdot}23$ (respectively). With these results we have constructed the conjectured envelope of all bounding neutral curves, and this is shown as a dotted line in figure 4. (This curve and the neutral curve for $P_m = 0{\cdot}2$ are indistinguishable, on the scale of figure 4, for the region $A^2 < 0{\cdot}28$.)

Whilst figures 1 and 2 essentially define the stability regions of our problem, *it*

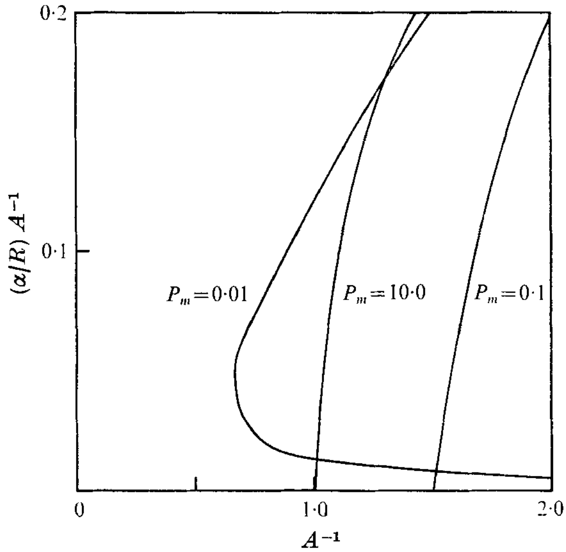

FIGURE 5. Neutral stability curves for various P_m. Stability is the region between the ordinate (α/AR) and the curve (for each case).

is of interest to consider one other diagram. In figure 5 we have plotted $(\alpha/R)\,A^{-1}$ against A^{-1} for various values of the magnetic Prandtl number. For fixed conductivity, kinematic viscosity and magnetic induction strength B_0 figure 5 may be interpreted as giving α_1 as a function of V, the characteristic velocity. Again, the features noted earlier are apparent and no further comment is necessary.

6. Concluding remarks

In the previous sections, we have considered in some detail the behaviour of a magnetic field on the stability of the plane interface between two conducting viscous fluids in relative motion. By use of the transformation introduced in § 3 the number of free parameters involved in our problem was reduced by one, thus considerably simplifying the subsequent investigation. Earlier workers in this field have apparently overlooked this transformation and, instead, reduced the number of free parameters by making assumptions concerning the nature of the original parameters (e.g. $R_m \ll 1$). The solution of the full problem – without such approximations being made – revealed several features not evident in the approximate solutions.

In interpreting our results it must be borne in mind that they are only applicable in the limit of small wavenumber: it is only in this limit that the use of a discontinuous velocity profile is justifiable.

To discuss the effect of the magnetic field on the stability of the flow it was found expedient to distinguish between the ranges $P_m < 1$ and $P_m > 1$. For $P_m \ll 1$ we found that the region of instability (and the value of A necessary to suppress the instability) increased with decreasing P_m. In the limiting case

$P_m = 0$ the field and flow become decoupled and the instability is always present, the critical Reynolds number being zero. This is, intuitively, to be expected.

At the other extreme (i.e. $P_m \gg 1$) the effect of the magnetic field is to stabilize the flow for $A > 1$. Again, this is the type of behaviour to be expected from a physical basis resting upon such ideas as the 'frozen-in' field, or considering the effect of the field as being equivalent to a surface tension at the interface (though this analogy must be applied with caution to a viscous flow).

In the intermediate range of magnetic Prandtl number (e.g. P_m of order unity) the interaction of the viscous and magnetic effects is considerably more complicated, and correspondingly it is difficult to obtain any valuable physical insight into the nature of these interactions.

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